Mathematical Aspects of Financial Risk

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November 22, 2013
WU Vienna
"Stochastics, Economics, and Architecture"

- Opening Conference of the Institute for Statistics and Mathematics on the

  New WU Campus

- in honor of the 65th Birthday of

  Helmut Strasser

International Conference Ars Conjectandi 1713-2013

October 15-16, 2013

Congress Center Basel, Switzerland

This conference will celebrate the **300th anniversary of the publication of Jacob Bernoulli's book Ars Conjectandi in 1713**. It is organised by the Swiss Statistical Society (SSS) and co-sponsored by the Bernoulli Society for Mathematical Statistics and Probability, the Institute of Mathematical Statistics (IMS) and the International Statistical Institute (ISI).

The conference will consist of keynote presentations of:

- David Aldous, Berkeley
- Peter Bühlmann, Zurich
- Louis Chen, Singapore
- Hans Föllmer, Berlin
- Tilmann Gneiting, Heidelberg
- Hans-Ruedi Künsch, Zurich
- Xiao-Li Meng, Cambridge
- Fritz Nagel, Basel
- Nancy Reid, Toronto
- Stephen Stigler, Chicago
- Edith Dudley Sylla, Raleigh
- Grace Wahba, Madison

The conference will be combined with the Swiss Statistics Meeting to be held on October 16-18, 2013, in Basel, Switzerland, celebrating the 25th anniversary of the SSS, the 15th anniversary of its section Official Statistics and the 10th anniversary of its sections Education and Research and Business and Industry.

Venue
Congress Center Basel (www.congress.ch), which is located right in the centre of Basel.

Further information and registration
www.statoo.ch/bernoulli13/

SWISS STATISTICAL SOCIETY www.stat.ch

Bernoulli Society for Mathematical Statistics and Probability

ims

ISSS since 1988

STATISTICS

T 3

Ars Conjectandi

Nr. 177

Vorlage: Universitätsbibliothek Basel, Kg VII 1.

JACOBI BERNOULLI,

Profeff. Bafil. & utriufque Societ. Reg. Scientiar. Gall. & Pruff. Sodal.

MATHEMATICI CELEBERRIMI,

ARS CONJECTANDI,

OPUS POSTHUMUM.

Accedit

TRACTATUS

DE SERIEBUS INFINITIS,

Et EPISTOLA Gallicè fcripta

DE LUDO PILÆ RETICULARIS.

BASILEÆ,

Impenfis THURNISIORUM, Fratrum.

cIↄ Iↄcc xiii.

I. Tractatus Hugenii de Ratiociniis in Ludo Aleae Cum Annotationibus Jacobi Bernoulli

II. Doctrina de Permutationibus & Combinationibus

III. Usus Praecedentis Doctrinae in variis Sortionibus & Ludis Aleae

Computation of "Valor Expectationis" of various gambles

~ pricing of contingent claims

(dice / coin tossing)

Christian Huygens:

Hoc autem ... utar
<u>fundamento:</u>

in aleae ludo tanti estimandum esse cujusque sortem seu <u>expectationem</u> ad aliquid obtinendum, <u>quantum si habeat,</u> possit denuo ad similem sortem sive expectationem pervenire, aequa conditione certans

= <u>cost of replication</u>

in the context of a fair game (in law)

Propositio I.

Si a vel b expectem, quorum utrumvis aeque facile mihi obtingere possit, expectatio mea dicenda est valere $(a+b)/2$

Proof*(!) by replication

in a fair game:

Stake $x := \frac{a+b}{2}$ in a fair game (coin toss), winner takes $2x$ and pays a to the loser

$$\tilde{X} = \begin{cases} b & \frac{1}{2} \\ a & \frac{1}{2} \end{cases}$$

---

* Ad hanc regulam non solum demonstrandum, verum etiam primitus eruendam posito x ...

Stephen Stigler:

"Huygens was the first published scientific probabilist as well as

<u>the first financial engineer</u> constructing hedges and derivative contracts in order to extend probability theory from common notions of <u>fairness in symmetric situations</u> to very different asymmetric markets"

$$P[A] = \frac{\#\text{ favorable cases}}{\#\text{ possible cases}}$$

"a priori":

quam in aleae ludis,
quos primi inventores ad
aequitem ipsis conciliandam
opera sic instituerunt,
ut <u>certi notique essent
numeri casuum</u>, at quos
sequi debet lucrum et damnum

Part IV (unfinished)

– tradens Usum et Applicationem Praecedentis Doctrinae in Civilibus, Moralibus & Oeconomicis

much wider scope

but:

$$P = ?$$

"a priori" does not apply:

"In caeteris enim plerisque vel a naturae operatione vel ab hominum arbitrium pendentibus effectis id neutiquam habet loco" "a posteriori"

"ex eventu in similibus exemplis multoties observato eruere"

↓

(weak) law of large numbers

("modus empiricus")

Leibniz (3.12.1703):

doubts the implicit stationarity assumptions

Agenda of Part IV

"Usus & Applicatio
in .... Oecономicis"

strong revival
last century,
in particular

in finance:

Agenda of Part IV

"Usus & Applicatio
in . . . . Oeconomicis"

strong revival
last century,
in particular
in finance:

A story of

- (partial) success
- (systemic) failure ?
- (mathematical) humility !

Language / Notation of Mathematics

more precisely:

mathematical language of randomness

= theory of probability

has entered / shaped / framed the discourse in finance (academia / industry) pioneered at MIT:

- P. A. Samuelson (1965):
  "Proof that properly anticipated prices fluctuate randomly"
  - martingale property
  - "efficient market hypothesis"
    (E. Fama, 1965, 1970)
    "strong/semi-strong/weak"

- rediscovery of
  "Théorie de la Spéculation"
  L. Bachelier (1900):

  Brownian Motion

Why

Brownian motion

(up to some transformation, Lévy process, ...)

?

~ "The Coin-Tossing View of Finance"

John Cassidy "Why Markets fall" (2009)

Simple heuristics:

many decisions (buy/sell) are made, more or less randomly / independently

(many coins are thrown ...)

" $\Longrightarrow$ "

Brownian motion should arise via

central limit theorem / invariance principle

manifestation of
Central Limit Theorem ?

Henri Poincaré (thesis report):

"Quand les hommes sont rapprochés, ils ne se décident plus au hasard et indépendamment les uns des autres; ils réagissent les uns sur les autres.

Des causes multiples entrent en action, et elles troublent les hommes, les entraînent à droite et à gauche, mais il y a une chose qu'elles ne peuvent détruire, ce sont les habitudes de

moutons de Panurge.

Et c'est cela qui se conserve.

cf. Alan Kirman (2010):
"The Economic Crisis is a Crisis for Economic Theory"

David Kreps

- John Bates Clark Medal 1989
- "Three esays on Capital Markets" (1979)

geometric Brownian motion
("Samuelson - Black - Scholes model")

= rational expectations equilibrium
for an economy of agents with

- preferences ~ power utility
- expectations ~ geometric BM
- demand computed on that basis

— the heroic view of rationality à la Arrow - Debreu - Radner
(cf. A. Kirman, H. Poincaré, ...)

Louis Bachelier :

"L'espérance mathématique du spéculateur est nulle"

i.e.

martingale property

= strong form of "efficient markets hypothesis"

+ continuous paths

+ stationary increments

$\Longrightarrow$ Brownian motion

P. Lévy

modern version :

Broad interdisciplinary consensus:

Price fluctuation
of a (liquid) financial asset

should be viewed as a

stochastic process

$$X_t(\omega)\ ,\quad t \geq 0$$

on some probability space

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$$

$\mathcal{P}$ = the

"historical" /

"physical" /

"objective"

<u>probability measure</u>

= ?

model ambiguity

"Knightian uncertainty"

de Finetti: does it even make sense?

typically

"<u>objectivist</u>"

interpretation / intuition:

- P exists
- P can be identified (partially) by statistical/econometric methods
- P should satisfy certain a priori constraints ("market efficiency"/ "absence of arbitrage")

Basic theoretical argument:

$\nexists$ "free lunch"

"$\Longleftrightarrow$" Kreps - Harrison

⋮

Delbaen - Schachermayer

$\exists$ "martingale measure"

$P^* \approx P$

i.e.

$$E^*[X_{t+h} - X_t \mid \mathcal{F}_t] = 0$$

up to localization

$\mathcal{P}^* =$ the (?)

"risk neutral" measure

= the market's linear

pricing operator

de Finetti: perfectly o.k.

in a "subjective" sense, as an

aggregate of subjective odds

for financial bets

"subject": financial market

(~ de Finetti's "you")

~ "representative investor"

cf. S. Ross

Thus : "no free lunch"

$$\Updownarrow$$

$\mathcal{P}^* :=$ all equivalent martingale measures $P^* \approx P$

$\neq \emptyset$

$\Downarrow$ Jacod, Yor, ...

(X_t) = "semimartingale"

= <u>stochastic integrator</u>

Dellacherie (→ Itô calculus)

= <u>Brownian motion</u> up to a random time change

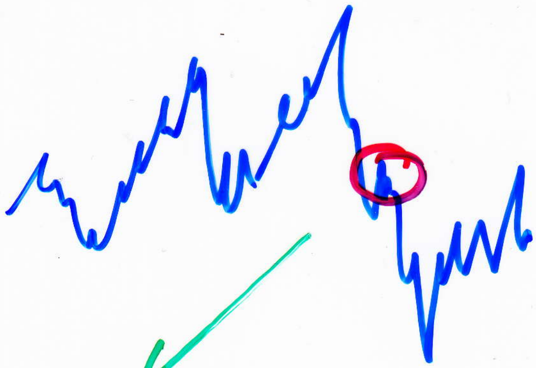
(W. Doeblin, ..., L. Dubins, ..., I. Monroe)
1940, 50's, 1972

So we have entered the field of

"Stochastic Analysis"

- a parallel world of great mathematical beauty and elegance
- in contrast to the messy reality of financial markets
- highly efficient for special purposes

($\neq$ prediction)



tick by tick, d assets:

$$K := \Big\{ \sum_{k=1}^{d} \xi_k \underbrace{\Delta X_k}_{\text{price increment of k-th asset}} \;\Big|\; \xi_k \in L^0(\Omega, \mathcal{F}_0, P) \Big\}$$

(NA) No arbitrage:

$$K \cap L^0_+(\Omega, \mathcal{F}, P) = \{0\}$$

(MM) $\exists$ martingale measure

$P^* \approx P$ (bounded density)

$$E^*[\Delta X_k \mid \mathcal{F}_0] = 0$$

To show:

$$(NA) \Longleftrightarrow (MM)$$

$$\Longrightarrow$$

crucial closure property
(to apply Hahn-Banach)

Lemma (W. Schachermayer)

(NA) $\Rightarrow$ $K - L^0_+(\Omega, \mathcal{F}, P)$
closed in $L^0(\Omega, \mathcal{F}, P)$

variants: Stricker, Kabanov-Kramkov

Lemma (A. Strasser):

$$\left.\begin{array}{ll} L & \mathcal{F}_0\text{-module} \\ C & \mathcal{F}_0\text{-cone} \\ L-C & \text{closed} \end{array}\right\} \text{ in } L^0(\Omega, \mathcal{F}, P)$$

$(K+L) \cap C = \{0\}$
$\Rightarrow K + L - C$ closed "

(Schachermayer: $L = \{0\}$, $C = L^0_+(\Omega, \mathcal{F}, P)$)
Stricker: $L = C = \{0\}$)

"Ad non solum demonstrandum, verum etiam primitus eruendam ...' "

mathematical / conceptual framework for

pricing / hedging

derivatives

"contingent claims"

= (non-linear) functionals A of underlying price fluctuation $(X_t)_{0 \le t \le T}$

"complete" case :

$$|\mathbb{P}^*| = 1$$

$$\Downarrow$$

"perfect replication"

$$H(\omega) = H_0 + \underbrace{\int_0^T \xi_t(\omega)\, dX_t(\omega)}_{\text{Itô integral}} \quad P\text{-a.s.}$$

= net gain from self-financing trading strategy

$\Longrightarrow$ unique arbitrage-free price

$\stackrel{!}{=}$ cost of replication

$= H_0 = E^*[H]$

Back to

computation of
expected values
of various
"gambles"
(= contingent claims)
in the context of a
fair game
~ the martingale measure
P^*

"Ars Conjectandi" (I, II)

Prediction

in terms of P

(drift, ...) :

does not intervene

matter !

standard setting in Financial Mathematics is probabilistic:

$\mathcal{P}$ vs. $\mathcal{P}^*$

"objective"/ "historical"/ "real world"/ measure

pricing measure = "the market's belief"

$\sim$

Girsanov

looking forward:

de Finetti (1931/37):

"Probability (P)
does not exist"

but:

prices (P^*) do!

via financial bets, based
on subjective degrees of
belief:

"You"

= the financial market

$$A = \left\{ \begin{array}{l} \text{Greek Bond} \\ \text{GR0114021463} \\ \text{does } \underline{\text{not}} \text{ default} \\ \text{at maturity } 20/08/2013 \end{array} \right\}$$

$$P[A] = \; ?$$

does the question make any "objective" sense ?

Where is the urn ?

However,

$P_t^*[A]$ à la de Finetti

"You" = financial market

or rather

$$E_t^*[\text{future cash flow}]$$

GRIECHENLAND 08/13 (A0TS58)

GR0114021463 | Frankfurt

65,50 EUR **0,00 %** 0,00 € 09:02 Uhr

Übersicht | Chart | Kurs | Wissen

Kursverlauf

Intraday | 1 Woche | 1 Monat | 3 Monate | 6 Monate | **1 Jahr** | 3 Jahre | 5 Jahre

as a

prediction scheme,

looking forward:

P vs. P^*

?	much more is known:

at each time t:

$$P^*_t := P^*[\,\cdot\,|\mathcal{F}_t]$$

∂ (traded claims with maturities $> t$)

= the market's implicit **prediction scheme** at time t

via prices of "plain vanilla" claims ($\to$ marginals Dupire) and more complex derivatives ($\to$ joint distributions)

— **consistent** (via arbitrage)

- across claims
- across future times $s > t$

But:

$\mathcal{P}^*_t$ = consistent view of the future at any fixed time t

may shift in a <u>non time-consistent</u> manner

$\Longleftrightarrow$ $\neq P^*[\,\cdot\,|\mathcal{F}_t]$ for one single $P^* \in \mathcal{P}^*$

(in an incomplete financial market)

$\sim$ "phase transition"

(1) Incompleteness

i.e., $|\wp^*| = \infty$,

Dynamics in $\wp^*$,

and

"Bubbles"

Eugene Fama:

"The word "bubble" drives me nuts"

Interview, Jan 13, 2010:

"I don't know what a bubble means. These words have become popular. I don't think they have any meaning".

"I didn't renew my subscription to 'The Economist' because they use the word bubble three times on every page. Any time prices went up and down — I guess that is what they call a bubble. People have become entirely sloppy."

Bubbles:

"irrational exuberance"

(A. Greenspan (1996):
"How do we know ...",
R. Shiller (2000))

vs.

"fundamentals"

Bubbles

"in the eye of the beholder":

Consider a liquid asset with

cumulative dividend process

$$0 \le D_t \nearrow D_\infty$$

and

price process

$$S_t \ge 0$$

(properly discounted / "relative")

$\mathcal{Q}^*$

defined in terms of

$$X_t := \underbrace{S_t + D_t}_{\text{"cum dividends" (discounted)}}$$

observed price $\qquad$ cumulative dividends

Every $P^* \in \mathcal{Q}^*$ supports observed prices in a speculative perspective:

$$S_t = \operatorname*{ess.sup}_{\tau \geq t} E^*\left[D_\tau - D_t + S_\tau \mid \mathcal{F}_t\right]$$

"option to sell"

cf. Harrison - Kreps (1978) "Speculative Investor Behavior with Heterogeneous Expectations"

But: different perceptions of "fundamental value"

$$E^*[D_\infty \mid \mathcal{F}_t]$$

for different $P^* \in \mathcal{P}^*$:

"Bubble"

$$\beta_t := S_t - E^*[D_\infty - D_t \mid \mathcal{F}_t]$$

$$= X_t - E^*[D_\infty \mid \mathcal{F}_t]$$

$$\begin{cases} = 0 & \text{for } P^* \in \mathcal{P}^*_{UI} \\ & (X \text{ unif. integrable}) \\ > 0 & \text{for } P^* \in \mathcal{P}^*_{NUI} \end{cases}$$

Typically:

$\mathcal{P}^*_{UI} \neq \emptyset$ and $\mathcal{P}^*_{NUI} \neq \emptyset$

i.e.

both views (Fama vs. Shiller)
coexist within $\mathcal{P}^*$

Generic examples:

Delbaen, Schachermayer (1998)

F., Biagini, Nedelcu (2013)
(stochastic volatility)

However:

Dichotomy for fixed $P^* \in \mathcal{P}^*$:

either no bubble at all,
or bubble is there right away
(supermartingale ≥ 0 cannot start in 0)

"Birth" of a bubble

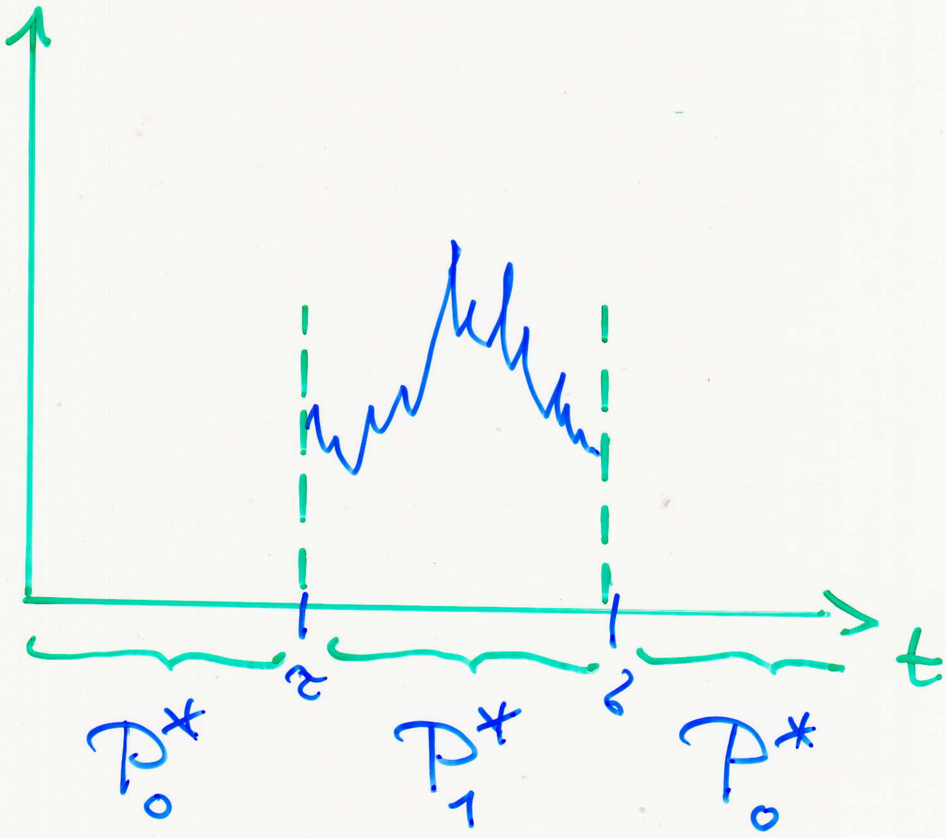
?

requires

dynamics in $\mathcal{P}^*$

Jarrow, Protter:

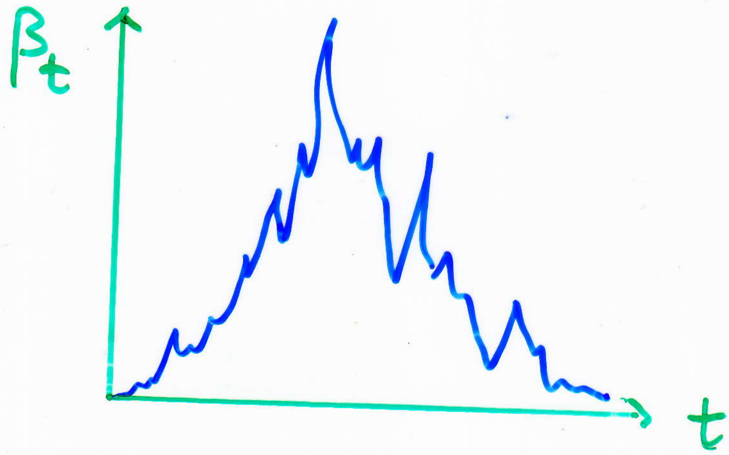
<u>sudden birth</u>
of a bubble
by sudden regime change:



Biagini, F., Nedelcu (2013):

flow in $\mathcal{P}^*$

from $\mathcal{P}^*_{NUI}$ to $\mathcal{P}^*_{UI}$



— an initial submartingale starting in 0 ("the birth"), then turning into a supermartingale falling back to 0 ("the decay")

— under a "sober" measure $P_0 \in \mathcal{P}^*_{NUI}$

Model / "Knightian"*

Ambiguity / Uncertainty

* in honor of

F. Knight:

"Risk, Uncertainty, and Profit"

Cornell / Yale: 1916 (21)

Risk =

"known unknowns"

$\sim \mathbb{P}$

(Knightian)
Uncertainty =

"unknown unknowns"

$\sim$ model ambiguity

$\underbrace{\mathcal{P}}_{\text{a whole class of "plausible" probabilistic models}}$

Possible approaches:

- argue "probability free", replacing P by its "support" (= a suitable space of paths)
- replace P by $\mathcal{P}$ (a whole class of "plausible" probabilistic models, focus on "worst case")

(2) Hedging without probability

Ω_σ = all continuous paths with fixed volatility pattern $\sigma^2(x,t)$

$$\sum_{t_i \le t} \left(X_{t_{i+1}}(\omega) - X_{t_i}(\omega)\right)^2$$

$$\xrightarrow{} \quad \langle X(\omega) \rangle_t = \int_0^t \sigma^2(X_s(\omega), s)\, ds$$

along dyadic partitions

(or: $\sigma^2(\cdot,\cdot) \in \Sigma$)

$\Rightarrow$ strictly pathwise replication
(or superreplication)

$$H(\omega) = H_0 + \underbrace{\int_0^T \xi_s(\omega)\, dX_s(\omega)}_{\text{strictly pathwise Itô-Integral}}$$

$\Rightarrow$ $\exists!$ martingale measure
Paul Lévy P^*_σ on Ω_σ :

$$H_0 = E^*_\sigma[H]$$

$$\left(= \sup_{\sigma \in \Sigma} E^*_\sigma[H] \right)$$
superreplication

"<u>plain vanilla</u>" options:

via "Calcul d'Itô sans probabilités" (A.F., 1981)

cf. also <u>Variance swaps</u> (A.F., A. Schied, 2013)

"<u>exotic" options</u>:

via strictly pathwise Malliavin calculus (R. Cont et al., 2009, 2011, ..)

---

in the same spirit:

"Finance without Probabilistic Prior Assumptions"

(Frank Riedel, 2011)

replace $(\Omega, \mathcal{F}, P)$ by polish space $(\Omega, \mathcal{F})$

Absence of arbitrage:

$$\nexists\, \xi : \quad \xi \cdot S(\omega) \geq 0 \quad \forall \omega \in \Omega$$

$$\{\underbrace{\xi \cdot S > 0}_{\text{open}}\} \neq \emptyset$$

$\Longleftrightarrow$ $\exists$ martingale measure P^* with full support

(in discrete time)

see also (Dublin 2013):

M. Soner, N. Touzi, ..

B. Bouchard, M. Nutz

M. Kupper, S. Drapeau, ...

B. Acciaio, W. Schachermayer, ...

(3) "**Robustify**":

$\mathcal{P}$ instead of P

Case study:

convex risk measures

(Artzner, Delbaen, Eber, Heath; Frittelli, Rosazza-Gianin; F., Schied; ...)

— Beyond "Value at Risk" and "law-invariance"

(cf. "stressed VaR", Basel III, ...)

Monetary risk as a capital requirement:

(regulatory perspective, cf. Basel II, III)

$$\rho(X) = \inf\{m \mid X + m \in \mathcal{A}\}$$

$\mathcal{A}$ = "acceptable positions"

$\mathcal{A}$ convex (diversification is not penalized)

$\Downarrow$ Fenchel-Moreau + "monetary"

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \Big(\underbrace{E_Q[-X]}_{\text{expected loss in model } Q} - \alpha(Q) \Big)$$

where

$\mathcal{Q}$:= a class of probability measures on $(\Omega, \mathcal{F})$

$$\alpha(Q) := \sup_{X \in \mathcal{A}} E_Q[-X]$$

— an explicit formalization of

model uncertainty

("Knightian" ")

"robust" view:

no probability measure is given a priori,
But:

probability measures do come in as

"stress tests" !

Research agenda

(1) Robust portfolio choice

find

"best" portfolio

under

$\underbrace{\text{financial}}_{\mathcal{P}^*}$ / $\underbrace{\text{risk constraints}}_{\text{convex risk measure}}$

$\uparrow$

$$\sup_{P^* \in \mathcal{P}^*} E^*[X] \overset{!}{\leq} c$$

= cost of (super-) replication

What does "best" mean?

preferences

$$X \succeq Y \iff U(X) \geq U(Y)$$

numerical representation

classical axioms of "rationality"

$\Downarrow$

von Neumann - Morgenstern
Savage
Aumann - Anscombe

$$U(X) = E_P[u(X)]$$

"expected utility"

(cf. D. Bernoulli)

more flexible axioms

(in view of "paradoxa", behavioral experiments, ...)

$\Downarrow$

Gilboa, Schmeidler (1989)

Maccheroni, Marinacci, Rustichini (2006)

$$U(X) = -\rho\left(u(X)\right)$$

$$= \inf_{Q \in \mathcal{Q}} \left(E_Q[u(X)] + \alpha(Q) \right)$$

ρ = convex risk measure

(Gilboa - Schmeidler: coherent)

solution of
optimization problem:

i) classical preferences, $\underbrace{|\mathcal{Q}^*| = 1}_{\text{"complete"}}$

$$X^* = (u')^{-1}\left(z \frac{dP^*}{dP}\right)$$

ii) classical preferences, $\underbrace{|\mathcal{Q}^*| = \infty}_{\text{"incomplete"}}$

<u>project P on $\mathcal{Q}^*$</u>
in terms of divergence $\sim u$
(= relative entropy if
u is exponential)

then take classical solution
w.r.t. P and $P_0^* :=$ projection

iii) $U(X) = \inf_{Q \in \mathcal{Q}} E_Q[u(X)]$

robust preferences

and $|\mathcal{P}^*| = \infty$:

<u>project $\mathcal{Q}$ on $\mathcal{P}^*$</u>

possibly extended to

$\bar{\mathcal{P}}^*$ = "extended martingale measures" on $\Omega \times (0, \infty]$ (predictable σ-field

– extending classical projection results of Csiszar, ...
(F., Gundel 2006)

② Dynamic Consistency

ρ_t = conditional risk measure, given the past

Consistency:

$$\rho_{t+1}(X) \geq \rho_{t+1}(Y) \Rightarrow \rho_t(X) \geq \rho_t(Y)$$

~ non-linear extension of martingale theory

~ backwards SDE (Peng, ...)

"bubble" effects in the dynamics of penalization (F.-Penner; Acciaio-F.-Penner)

"Spatial" risk measures:

Local specification,
Asymptotics,
(~ thermodynamical limit)
Phase transition

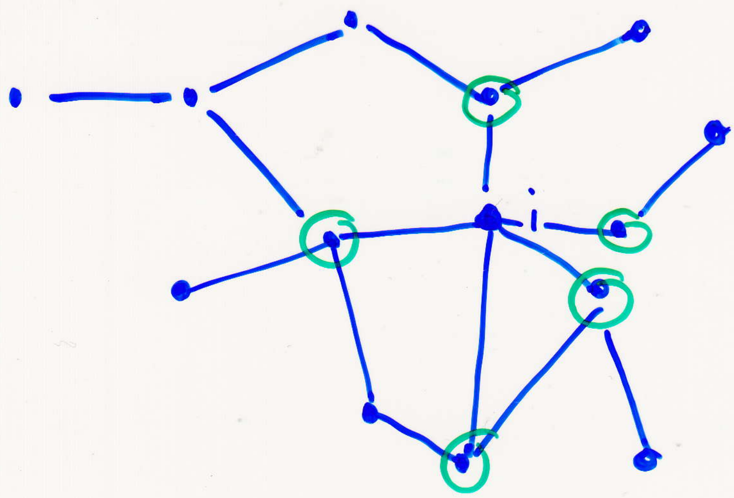
~ Gibbs measures

cf.

C. Cotar, C. Klüppelberg, A.F.
(work in progress)
A.F.: the locally law-invariant case

I = a large set of "sites"

$\sim$ a network of financial institutions



$N(i)$ = "neighbors" of i

$\rho_{\{i\}}(\omega, X)$ — assess the risk at site i conditionally!

"environment" = situation on $I - \{i\}$ (or $N(i)$)

Problem A:

<u>consistent aggregation</u>

from single nodes $i \in I$
to finite subsets $V \subseteq I$:

$$\rho_V \left(X \,\middle|\, \begin{matrix} \text{environment} \\ \text{outside of } V \end{matrix} \right)$$

?

probabilistic case: $\Longleftrightarrow$

Gibbsian description
in terms of interaction potentials

Problem B: $|I| = \infty$
"large" network

Given a local specification

$\rho_V(\cdot \mid \text{environment})$ ($V \subseteq I$ finite),

what is the structure of

$\mathcal{R} :=$ all global convex risk measures ρ that are consistent with the local specification, i.e.,

$$\rho = \rho(-\rho_V) \quad \forall V$$

non-uniqueness = "phase transition"

~ one aspect of "systemic risk"

<u>In general:</u> ?

some partial results
work in progress with
C. Cotar, C. Klüppelberg

<u>locally law-invariant case:</u>
(A.F., 2013)

$g_V(\cdot \mid \cdot)$ only depends on

$\pi_V(\cdot \mid \cdot)$

= conditional distribution of
a "<u>Gibbs measure</u>"

$\mathcal{P}$:= all random fields P
consistent with (π_V)

A: The risk measures $\rho_V(\cdot|\cdot)$ must be "entropic", i.e.,

$$\rho_V(x|\eta) = \frac{1}{\beta(\eta)} \log \int e^{-\beta(\eta) X(\cdot,\eta)} \, d\pi_V(\cdot|\eta)$$

η = configuration outside of V

(penalization in terms of conditional relative entropy)

$\beta(\cdot)$ measurable with respect to "tail field"

$$\hat{\mathcal{F}} := \bigcap_{V \text{ finite}} \mathcal{F}_{V^c}$$

$\mathcal{B}$:

$\mathcal{R}$:= all <u>global</u> convex risk measures consistent with (ρ_V)

$= \{ \hat{\rho}(-\rho_\infty) \mid \hat{\rho} =$ convex risk measure on tail field $\hat{\mathcal{F}} \}$

where

$$\rho_\infty = \frac{1}{\beta(\cdot)} \log \int e^{-\beta X(\omega)} \pi_\infty(d\omega \mid \eta)$$

common conditional distribution of all Gibbs measures $P \in \mathcal{P}$ w.r.t. $\hat{\mathcal{F}}$ (F., Dynkin)

In particular:

$|\alpha| > 1$ "phase transition"

$\Longleftrightarrow$ true tail-dependence of $\beta(\cdot)$

and/or

probabilistic phase transition $|\rho| > 1$

Beyond (local) law-invariance:

- largely uncharted territory

But:

possible message:

<u>local</u>

(even conditional, vs. just marginal)

risk analysis

may not suffice to capture all sources of the aggregate / "<u>systemic" risk</u>

– in non-linear analogy to the probabilistic analysis of phase transitions

In contrast to Turner Review:

"Knightian Uncertainty"

is not orthogonal to "sophisticated maths", but a

<u>rich source</u>

of mathematical problems!

Thanks

for your attention

and

Best wishes

- to the Institute
- to

Helmut Strasser!